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QUASIFACTORS OF ERGODIC SYSTEMS WITH POSITIVE ENTROPY

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ABSTRACT

The relation between the two notions, quasifactors and joinings, is investigated and the notion of a joining quasifactor is introduced. We clarify the close connection between quasifactors and symmetric infinite selfjoinings which arises from de Finetti–Hewitt–Savage theorem. Unlike the zero-entropy case where quasifactors seems to preserve some properties of their parent system, it is shown that any ergodic system of positive entropy admits all ergodic systems of positive entropy as quasifactors. A restricted version of this result is obtained for joining quasifactors.

0. Introduction

For a measure preserving transformation $(X, \mathcal{X}, \mu, T)$, a factor system $(Y, \mathcal{Y}, \nu, S)$ with a factor map π: $X \to Y$ can be viewed as the T-invariant subalgebra $\pi^{-1}(\mathcal{Y}) \subset \mathcal{X}$. One can also describe the factor $(Y, \mathcal{Y}, \nu, S)$ as a measure preserving transformation on the space $M(X)$ of probability measures on X as follows. Disintegrate the measure μ along the fibers of $\pi^{-1}(\mathcal{Y})$,

$$(*) \qquad \mu = \int_Y \mu_y d\nu(y),$$

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and observe that the T-invariance of μ implies that $T\mu_y = \mu_{Sy}$. Denoting by ϕ: $Y \to M(X)$ the map $\phi(y) = \mu_y$ and letting $\nu^* = \phi_*(\nu)$, we see that ϕ: $(Y, \mathcal{Y}, \nu, T) \to (M(X), \nu^*, T)$ is an isomorphism. The connection with μ is given by $(*)$ which says that μ is the **barycenter** of ν^*. A general **quasifactor** of $(X, \mathcal{B}, \mu, T)$ is any T-invariant measure on $M(X)$ whose barycenter is μ. This notion was introduced in [G] and further studied in [GW], where it was shown that, like factors, quasifactors inherit some dynamical properties. For example, zero entropy and distality (which implies zero entropy) are preserved under a passage to a quasifactor.

A joining λ of two systems $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$, i.e., a $T \times S$-invariant probability on $X \times Y$ which projects onto μ and ν respectively, similarly gives rise to a quasifactor of $(X, \mathcal{X}, \mu, T)$ by disintegrating λ over ν:

$$\lambda = \int_Y \mu_y \times \delta_y d\nu(y)$$

(and, of course, symmetrically, a quasifactor of $(Y, \mathcal{Y}, \nu, S)$). In fact every quasifactor of $(X, \mathcal{X}, \mu, T)$ can be obtained this way from a joining, but in general the joining λ carries more information than there is in the corresponding quasifactor. Moreover, even when the quasifactor we start from is ergodic, this need not be the case with the associated joining.

In this work we investigate the relation between the two notions, quasifactors and joinings, and show that, unlike the zero entropy case where quasifactors seems to preserve some properties of their parent system, any ergodic system of positive entropy admits all ergodic systems of positive entropy as quasifactors.

More specifically, the paper is arranged as follows. In the first section we clarify the close connection between quasifactors and symmetric infinite selfjoinings which arises from the de Finetti–Hewitt–Savage theorem (Theorem 1.2). As a corollary we retrieve our zero entropy results on quasifactors (Corollary 1.3). (The recent comprehensive work of Lemańczyk, Parreau and Thouvenot on a family of Gaussian automorphisms (the "GAG"s) [LPT] contains related results on joinings and disjointness; see also [GTW].) In section 2 the notion of a "joining quasifactor" is introduced. In section 3 we prove the above assertion about positive entropy systems (Theorem 3.6). We make use of a result of Smorodinsky and Thouvenot [ST], showing that any positive entropy ergodic system is spanned by three Bernoulli factors. In the last section we prove the same theorem for joining quasifactors (Theorem 4.1). For this we develop a variant of [ST] involving symmetric joinings of Bernoulli shifts the proof of which is based on the beautiful theorem of J. Kieffer [K] on zero error coding. However, we can only prove this stronger result up to finite to one extensions and for systems satisfying the weak Pinsker property and having finite entropy.

1. Quasifactors and infinite order symmetric selfjoinings

Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an ergodic measure preserving dynamical system on a standard Borel space $(X, \mathcal{X})$. Let $M = M(X)$ be the space of probability measures on X with the Borel structure determined by the maps $\mu \mapsto \mu(A)$ for a fixed $A \in \mathcal{X}$. Recall that κ, a probability measure on the space $M = M(X)$, is a *quasifactor* of the system $\mathbf{X}$ if it is T-invariant, and satisfies the *barycenter equation*:

$$\int_{M(X)} \theta \, d\kappa(\theta) = \mu.$$

This equation means that — choosing any compact topology on X compatible with its Borel structure — for every continuous function $f \in C(X)$,

$$\int_M \int_X f(x) \, d\theta(x) \, d\kappa(\theta) = \int_X f(x) \, d\mu(x).$$

It turns out that this definition does not depend on the particular choice of compact metric topology on X, [G]. The quasifactor κ is **ergodic** if the system (M, κ) is ergodic. We let $Q(X, \mu)$ denote the set of quasifactors of $\mathbf{X}$ and $Q_e(X, \mu)$ the subset of ergodic quasifactors. It is sometimes convenient to dissociate the quasifactor from the original system by writing W for $M(X)$, with the bijection $w \leftrightarrow \theta_w$. In this notation the barycenter equation becomes

$$\int_W \theta_w \, d\kappa(w) = \mu$$

and we write (W, κ, T) for the system $(M(X), \kappa, T)$.

Given $\kappa \in Q(X, \mu)$ set

$$\kappa' = \int_{M(X)} (\theta \times \delta_\theta) d\kappa(\theta),$$

a probability measure on $X' := X \times M(X)$ which is a joining of μ and κ. For a positive integer $k \geq 1$ we let

$$\kappa^{(k)} = \int_{M(X)} (\theta \times \theta \times \cdots \times \theta) \times \delta_\theta d\kappa(\theta),$$

where $\theta \times \theta \times \cdots \times \theta$ is a k-fold product, and

$$\kappa^{(\infty)} = \int_{M(X)} (\cdots \theta \times \theta \times \theta \cdots) \times \delta_\theta d\kappa(\theta).$$

We let

$$\bar{\kappa}^{(k)} = \int_{M(X)} \theta \times \theta \times \cdots \times \theta d\kappa(\theta) \quad \text{and} \quad \bar{\kappa}^{(\infty)} = \int_{M(X)} (\cdots \theta \times \theta \times \theta \cdots) d\kappa(\theta)$$

be the corresponding projections onto X^k and X^∞, respectively. The measures $\bar{\kappa}^{(k)}$ and $\bar{\kappa}^{(\infty)}$ are selfjoinings of the system (X, μ). In particular, $\bar{\kappa}^{(1)} = \bar{\kappa}' = \mu$ and $\bar{\kappa}^{(2)} = \int_{M(X)} \theta \times \theta d\kappa(\theta)$.

We will next consider the close connection between quasifactors and symmetric infinite selfjoinings which arises from the de Finetti–Hewitt–Savage theorem.

Let X be a compact metric space. As usual $X^{\mathbb{Z}}$ is the infinite product space and we let $\mathfrak{S}$ be the group of permutations of coordinates in $X^{\mathbb{Z}}$ which fix all but finitely many of them. Let $M(X)$ be the compact metric Bauer simplex of all probability Borel measures on X and let $\mathcal{Q} = M(M(X))$ be the space of probability Borel measures on $M(X)$. Let $\mathcal{J} \subset M(X^{\mathbb{Z}})$ be the convex closed subset of $M(X^{\mathbb{Z}})$ consisting of all probability measures on $X^{\mathbb{Z}}$ which are invariant under $\mathfrak{S}$; these measures are called **symmetric**. We set

$$(**) \qquad \begin{gathered} \phi\colon \mathcal{Q} \to \mathcal{J}, \\ \kappa \mapsto \phi(\kappa) = \lambda = \bar{\kappa}^{(\infty)} = \int_{M(X)} (\cdots \theta \times \theta \times \theta \cdots) d\kappa(\theta). \end{gathered}$$

Clearly ϕ is a continuous affine map of $\mathcal{Q}$ into $\mathcal{J}$. The de Finetti–Hewitt–Savage theorem, [HS], in the version proved by L. Dubins (see [D]), states that this map is an affine homeomorphism of $\mathcal{Q}$ onto $\mathcal{J}$. In particular, as in the Bauer simplex $\mathcal{Q}$ the set of extreme points is the closed set

$$\{\delta_\theta : \theta \in M(X)\};$$

it follows that in $\mathcal{J}$, the set of extreme points is the closed set

$$\{\phi(\delta_\theta) : \theta \in M(X)\} = \{\hat{\theta} = \cdots \theta \times \theta \times \theta \cdots : \theta \in M(X)\},$$

and that $(**)$ is the unique Choquet representation of an element λ of the simplex $\mathcal{J}$ as an integral over the set of extreme points, or, in other words, its $\mathfrak{S}$ ergodic decomposition.

THEOREM 1.1 (de Finetti–Hewitt–Savage): *The map ϕ: $\mathcal{Q} \to \mathcal{J}$ is an affine homeomorphism.*

Let now $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ be an ergodic system with X a compact metric space and $\{T^n : n \in \mathbb{Z}\}$ acting on X as a group of homeomorphisms. Let T act on

$X^{\mathbb{Z}}$ by the diagonal action. We denote by $J^{\infty}_{\rm sym}(\mathbf{X})$ the set of all $\mathfrak{S}$-symmetric T-invariant probability measures on $X^{\mathbb{Z}}$ with marginal μ on the 0 (hence also every $n \in \mathbb{Z}$) coordinate. Clearly, for every $\kappa \in Q(X,\mu)$ the corresponding infinite joining $\bar{\kappa}^{(\infty)}$ is in $J^{\infty}_{\rm sym}(\mathbf{X})$.

THEOREM 1.2:

(1) *Every quasifactor κ of the system (X,μ) is canonically isomorphic to a factor of the infinite order, symmetric, selfjoining $(X^{\mathbb{Z}}, \bar{\kappa}^{(\infty)}) \in J^{\infty}_{\rm sym}(\mathbf{X})$ of (X,μ). The factor map α: $(X^{\mathbb{Z}}, \bar{\kappa}^{(\infty)}) \to (W,\kappa) = (M(X),\kappa)$ is the factor map which corresponds to the σ-algebra of $\mathfrak{S}$-invariant sets; i.e., to the $\mathfrak{S}$-ergodic decomposition. Thus, denoting $\zeta_w = \cdots\theta_w \times \theta_w \times \theta_w \cdots$, the $\mathfrak{S}$-ergodic decomposition*

$$\bar{\kappa}^{(\infty)} = \int_W \zeta_w d\kappa(w)$$

coincides with the disintegration of $\bar{\kappa}^{(\infty)}$ over κ with respect to α, and for κ a.e. w, $\zeta_w(\alpha^{-1}(w)) = 1$.

(2) *The map ϕ: $\kappa \mapsto \bar{\kappa}^{(\infty)}$, ϕ: $Q(X,\mu) \to J^{\infty}_{\rm sym}(\mathbf{X})$ is an affine homeomorphism onto.*

Proof: (1) Fix a compact metric topology on X and observe that $Q(X,\mu)$ is a closed convex subset of $\mathcal{Q}$ and that $J^{\infty}_{\rm sym}(\mathbf{X})$ is a closed convex subset of $\mathcal{J}$. Moreover, clearly ϕ: $\mathcal{Q} \to \mathcal{J}$ maps $Q(X,\mu)$ into $J^{\infty}_{\rm sym}(\mathbf{X})$. If $\lambda \in J^{\infty}_{\rm sym}(\mathbf{X})$, then λ as an element of $\mathcal{J}$ has a unique representation $(**)$. Applying T to $(**)$ we get

$$\begin{aligned} T\lambda &= \int_{M(X)} (\cdots T\theta \times T\theta \times T\theta \cdots) d\kappa(\theta) \\ &= \int_{M(X)} (\cdots \theta \times \theta \times \theta \cdots) dT\kappa(\theta). \end{aligned}$$

On the other hand

$$T\lambda = \lambda = \int_{M(X)} (\cdots \theta \times \theta \times \theta \cdots) d\kappa(\theta),$$

and the uniqueness of the representation implies that $T\kappa = \kappa$. By projecting the representation $(**)$ of λ on, say, the zero coordinate, we see that κ satisfies the barycenter equation and we conclude that $\kappa \in Q(X,\mu)$ and that $\phi(\kappa) = \lambda$. Thus ϕ: $\mathcal{Q} \to \mathcal{J}$ maps $Q(X,\mu)$ onto $J^{\infty}_{\rm sym}(\mathbf{X})$.

It is now clear that the map α: $(X^Z, \bar{\kappa}^{(\infty)}, T) \to (M(X), \kappa, T)$ which corresponds to the $\mathfrak{S}$ ergodic decomposition

$$\bar{\kappa}^{(\infty)} = \int_W \zeta_w d\kappa(w)$$

is indeed a T homomorphism. In particular, κ a.e. the product measures, ζ_w are mutually singular and $\zeta_w(\alpha^{-1}(w)) = 1$.

(2) This follows from the fact that $\phi\colon \mathcal{Q} \to \mathcal{J}$ is an affine homeomorphism. ∎

COROLLARY 1.3: *Let P be a property of ergodic systems which is preserved by infinite ergodic selfjoinings as well as factors. Then every quasifactor of a system with property P also has property P. In particular, distality and zeroentropy are two examples of such properties.*

For more details see [GW]. The works [LPT] and [GTW] deal with the same subject from a different viewpoint.

2. Joining quasifactors

Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$and $\mathbf{Y} = (Y, \mathcal{Y}, \nu, S)$be ergodic measure preserving dynamical systems. Let $\lambda \in J_e(\mu, \nu)$ be an ergodic joining and let

$$\lambda = \int_Y (\lambda_y \times \delta_y)\, d\nu(y)$$

be its disintegration over ν. Let $\phi\colon Y \to M = M(X)$ be the measurable map associated with this disintegration, $\phi\colon y \mapsto \lambda_y$ from Y to the space of probability measures on X and let $\kappa := \nu^* = \phi_*(\nu)$ be the image of ν under ϕ. The $T \times S$ invariance of λ and the uniqueness of disintegration imply that ν-a.e. $T\lambda_y = \lambda_{Sy}$. Thus ϕ is a homomorphism of the system $(Y, \mathcal{Y}, \nu, S)$onto the system $\mathbf{M} = (M(X), \kappa, T)$, where we use T also for the map induced by T on $M(X)$.

We have the barycenter equation for κ:

$$\int_{M(X)} \theta\, d\kappa(\theta) = \int_Y \lambda_y\, d\nu(y) = \mu.$$

Thus the system $\mathbf{M} = (M(X), \kappa, T)$ is a factor of $(Y, \mathcal{Y}, \nu, S)$ and a quasifactor of (X, μ, T). With the factor map $\phi\colon Y \to M = M(X)$, we associate the disintegration of ν over $\nu^* = \kappa$:

$$\nu = \int_M \nu_\theta d\tilde{\nu} = \int_M \nu_{\lambda_y}\, d\nu^*(\lambda_y) = \int_M \nu_{y^*}\, d\nu^*(y^*),$$

where for $y \in Y$ we let $y^* = \lambda_y = \phi(y)$.

Two natural joinings now arise:

$$\kappa' = \int_M (\theta \times \delta_\theta)\, d\kappa(\theta) = \int_Y (\lambda_y \times \delta_{\lambda_y})\, d\nu(y),$$

on $X' := X \times M$, and

$$\lambda' = \int_Y (\lambda_y \times \delta_{\lambda_y} \times \delta_y)\, d\nu(y),$$

on $X' \times Y$.

Clearly the map ψ: $(x, \lambda_y, y) \mapsto (x, y)$ from $X' \times Y$ to $X \times Y$ is 1-1 and equivariant, and applying ψ to the definition of λ' we get

$$\begin{aligned}\psi(\lambda') &= \psi\Big(\int_Y (\lambda_y \times \delta_{\lambda_y} \times \delta_y)\, d\nu(y)\Big)\\ &= \int_Y (\lambda_y \times \delta_y)\, d\nu(y) = \lambda,\end{aligned}$$

so that ψ is a canonical isomorphism of $(X' \times Y, \lambda')$ onto $(X \times Y, \lambda)$.

But we also see that

$$\begin{aligned}\lambda' &= \int_Y (\lambda_y \times \delta_{y^*} \times \delta_y)\, d\nu(y)\\ &= \int_M \int_Y (\lambda_y \times \delta_{y^*} \times \delta_y)\, d\nu_{y^*}(y)\, d\kappa(y^*)\\ &= \int_M (\lambda_{y^*} \times \delta_{y^*} \times \nu_{y^*})\, d\kappa(y^*)\\ &= \kappa' \underset{\kappa}{\times} \nu,\end{aligned}$$

so that finally

$$(X' \times Y, \lambda') \cong (X' \underset{M}{\times} Y, \kappa' \underset{\kappa}{\times} \nu) \cong (X \times Y, \lambda).$$

As a corollary of this discussion we see that every ergodic joining λ of two ergodic systems (X, μ) and (Y, ν) is in fact isomorphic to the relatively independent joining $\lambda' = \kappa' \underset{\kappa}{\times} \nu$ of a canonical extension (X', κ') of (X, μ), with (Y, ν), over the common factor (M, κ). Of course, this relative product becomes "trivial" when the map $y \mapsto \lambda_y$ is an isomorphism of $\mathbf{Y} = (Y, \nu)$ onto the quasifactor $\mathbf{M} = (M, \nu^*) = (M, \kappa)$, in which case also $(X \times Y, \lambda)$ is isomorphic to (X', κ') (as we shall see below, Proposition 2.2, this is exactly the case when we call $\mathbf{M}$ a joining quasifactor). We have shown:

PROPOSITION 2.1: *Let $\mathbf{X} = (X, \mathcal{X}, \mu, T)$ and $\mathbf{Y} = (Y, \mathcal{Y}, \nu, S)$ be ergodic measure preserving dynamical systems, and λ an ergodic joining. Form the dynamical systems $\mathbf{M}$ and $\mathbf{X}'$ as above. Then $\mathbf{M}$ is a quasifactor of $\mathbf{X}$, a factor of $\mathbf{Y}$, and* *in the commutative diagram:*

$$\begin{array}{ccccc} & & (X \times Y, \lambda) & & \\ & \swarrow & & \searrow & \\ (X', \kappa') & & & & (Y, \nu) \\ \swarrow & \searrow & & \swarrow & \searrow \\ (X, \mu) & & (M, \kappa) & & (Y, \nu), \end{array}$$

we have $\lambda = \kappa' \underset{\kappa}{\times} \nu$.

Definition: Let $(X, \mathcal{X}, \mu, T)$be an ergodic system and κ an ergodic quasifactor of (X, μ). We say that κ is a **joining quasifactor**, jqf for short, if the joining

$$\kappa' = \int_M (\theta \times \delta_\theta) \, d\kappa(\theta)$$

of the systems (X, μ) and (M, κ) is ergodic. We denote the collection of joining quasifactors of (X, μ) by $Q_J(X, \mu)$.

PROPOSITION 2.2: *Let* $\mathbf{X} = (X, \mathcal{X}, \mu, T)$*be an ergodic system.*

(1) *The ergodic quasifactor* κ *of the system* $\mathbf{X}$ *is a jqf iff there exists an ergodic system* $\mathbf{Y}$ $=(Y, \mathcal{Y}, \nu, S)$*and an ergodic joining* λ *of* (X, μ) *and* (Y, ν) *such that* $\kappa = \nu^*$*; i.e., if*

$$\lambda = \int_Y (\lambda_y \times \delta_y) \, d\nu(y)$$

is the disintegration of λ *over* ν, ϕ: $Y \to M = M(X)$, ϕ: $y \mapsto \lambda_y$ *the associated map and* $\nu^* = \phi(\nu)$, *then* $\kappa = \nu^*$.

(2) *The ergodic system* $\mathbf{Y}$ $=(Y, \mathcal{Y}, \nu, S)$*appears as a jqf of* $\mathbf{X}$ *iff there is an ergodic joining* $\lambda \in J_e(\mu, \nu)$ *such that in the disintegration*

$$\lambda = \int_Y (\lambda_y \times \delta_y) \, d\nu(y),$$

the map $y \mapsto \lambda_y$ *is 1-1.*

Proof: (1) If κ is a jqf, we take $(Y, \nu) = (M(X), \kappa)$ and

$$\lambda = \kappa' = \int_M (\theta \times \delta_\theta) \, d\kappa(\theta),$$

which by assumption is ergodic. Conversely, if an ergodic system $(Y, \mathcal{Y}, \nu, S)$and an ergodic joining λ of (X, μ) and (Y, ν) are given, Proposition 3.1 shows that

$\kappa := \phi(\nu) = \nu^*$ is an ergodic quasifactor of (X, μ) and that κ' is ergodic. Part (2) is clear. ∎

Definition: Let (X, μ) be an ergodic system and (W, κ) a jqf. We say that κ is:

(1) **ergodically embedded** if the extension $(X \times W, \kappa') \to (W, \kappa)$ is an ergodic extension,

(2) **weakly mixing embedded** if this extension is a weakly mixing extension,

(3) **mixingly embedded** if this extension is a mixing extension,

(4) **continuously embedded** if κ a.e. θ_w is a continuous measure (has no atoms) and **finitely embedded** if there exists a positive integer r such that κ a.e. θ_w is an equidistributed measure on a set of r points. In the latter case we say that κ is of **type** r.

Remarks: (1) As one can easily see, for a dynamical system (X, μ) with factor π: $(X, \mu) \to (Y, \nu)$, the system (X, μ) is ergodic iff both the system (Y, ν) and the extension π are ergodic. It thus follows that a qf is ergodically embedded iff it is a joining quasifactor.

(2) Let $f(x, w) = \theta_w(x)$, then κ' a.e.

$$f(Tx, Tw) = \theta_{Tw}(Tx) = T(\theta_w)(Tx) = \theta_w(T^{-1}Tx) = f(x, w).$$

Thus the ergodicity of κ' implies that either $f \equiv 0$, i.e., κ is continuously embedded, or $f \equiv 1/r$, i.e., κ is finitely embedded of type r.

(3) Mixingly embedded quasifactors are used in Nevo and Zimmer [NZ] and in Furstenberg and Glasner [FG].

3. Quasifactors of positive entropy systems

In [GW] we have shown that every quasifactor of a zero entropy system is itself of zero entropy. Since zero-entropy systems are disjoint from every completely positive entropy system, no nontrivial zero-entropy system can appear as a quasifactor of a Bernoulli system or, more generally, of a completely positive entropy system. In the rest of the paper we investigate quasifactors of positive entropy systems. We begin with two lemmas concerning lifting of quasifactors.

The first lemma says that a quasifactor of a factor is also a quasifactor of the original system. On the other hand, not every factor of a quasifactor is a quasifactor of the original system. Indeed, as we will see later (Theorem 3.6), every ergodic system of positive entropy, say (Y, ν), admits every ergodic system of positive entropy, say (X, μ), as a quasifactor. If we take in this statement (Y, ν) to be a Bernoulli system and $(X, \mu) = (Y \times Z, \nu \times \eta)$ with (Z, η) any zero-entropy system, we see that the factor (Z,η) of the quasifactor (X,μ) of the system (Y,ν) cannot appear as a quasifactor of (Y,ν).

LEMMA 3.1: *If $(X,\mu)\to(Y,\nu)$ is a homomorphism of ergodic dynamical systems, then every quasifactor of (Y,ν) is also a quasifactor of (X,μ).*

Proof: Let

$$\mu=\int_Y \mu_y \, d\nu(y)$$

be the disintegration of μ over ν. Now the map $L\colon M(Y)\to M(X)$ defined by

$$L(\theta)=\int_Y \mu_y \, d\theta(y)$$

is an affine isomorphism which satisfies $L(T\theta)=TL(\theta)$, $\forall\theta\in M(Y)$ and $L(\nu)=\mu$. It follows easily that via L every quasifactor of (Y,ν) lifts to an isomorphic quasifactor of (X,μ). These are the quasifactors $\rho=L_*(\kappa)$, $\kappa\in Q_e(Y,\nu)$ of (X,μ) that are supported on $L(M(Y))$. ∎

LEMMA 3.2: *Let $\pi\colon(X,\mu)\to(Y,\nu)$ be a homomorphism of ergodic dynamical systems and $\kappa\in Q_e(Y,\nu)$ an ergodic quasifactor. Let $L\colon M(Y)\to M(X)$ be as in the previous lemma and $\rho\in Q_e(X,\mu)$ be given by $\rho=L_*(\kappa)$. For convenience, let us write $(M(Y),\kappa):=(W,\kappa)$ with θ_w as a typical element of $M(Y)$. Denoting as usual by κ' and ρ' the joinings*

$$\kappa'=\int_W \theta_w\times\delta_w \, d\kappa(w) \quad\text{and}\quad \rho'=\int_W L(\theta_w)\times\delta_w \, d\kappa(w),$$

we have

$$\rho'=\mu\underset{\nu}{\times}\kappa'.$$

Proof: Disintegrating $\mu\underset{\nu}{\times}\kappa'$ over κ' we clearly have

$$\mu\underset{\nu}{\times}\kappa'=\int_{Y\times W}\mu_y\times\delta_w \, d\kappa'(y,w).$$

Using the definition of κ' we get

$$\begin{aligned}\mu\underset{\nu}{\times}\kappa' &=\int_{Y\times W}\mu_y\times\delta_w \, d\kappa'(y,w)\\ &=\int_W\int_Y \mu_y\times\delta_w \, d(\theta_w\times\delta_w)(y,w)\, d\kappa(w)\\ &=\int_W\Big(\int_Y \mu_y \, d\theta_w\Big)\times\delta_w \, d\kappa(w)\\ &=\int_W L(\theta_w)\times\delta_w \, d\kappa(w)=\rho'. \qquad ∎\end{aligned}$$

COROLLARY 3.3: *With notations as in the previous lemma, if κ is a joining quasifactor of (Y,ν) and either $(X,\mu) \to (Y,\nu)$ or $(Y \times W,\kappa') \to (Y,\nu)$ is a weakly mixing extension, then the quasifactor ρ, the lift of κ to $M(X)$, is also a joining quasifactor.*

PROPOSITION 3.4: *Every weakly mixing system (X,μ) of positive entropy h is a weakly mixing extension of a Bernoulli system (Y,ν) with entropy h', for every $0 < h' \le h$.*

Proof: Use Sinai's theorem to find a Bernoulli factor Y' of entropy h'. If the extension $X \to Y'$ is not weakly mixing, use the relative Furstenberg–Zimmer theorem to find a maximal distal extension $Y \to Y'$ so that the extension $X \to Y$ is weakly mixing (see [F]). Now every distal extension is a tower of isometric extensions and inverse limits. By Rudolph's theorem, [R], a weakly mixing system which is an isometric extension of a Bernoulli system is Bernoulli, and the inverse limit of Bernoulli systems is also Bernoulli. Thus Y is a Bernoulli factor of X. ∎

Given a dynamical system $\mathbf{Y} = (Y,\mathcal{Y},\nu,T)$ and a positive integer $q \ge 2$, we let S_q be the finite group of all permutations of coordinates on $Y^q = Y \times Y \times \cdots \times Y$. Let $\sigma\colon Y^q \to Y^{(q)} = Y^q/S_q$ be the quotient map. We denote by $\langle y_1, y_2, \ldots, y_q\rangle$ a typical point of $Y^{(q)}$.

LEMMA 3.5: *Let $\mathbf{Y} = (Y,\mathcal{Y},\nu,T)$ be an ergodic system and let $(Y \times Y \times \cdots \times Y, \lambda)$ be a q-fold selfjoining of (Y,ν), q a positive integer or ∞.*

(1) *Let $\{\alpha_1,\alpha_2,\ldots,\alpha_q\}$ be a set of q distinct positive real numbers with sum 1 and define a map*

$$\phi\colon Y \times Y \times \cdots \times Y \to M(Y),$$

$$\phi\colon (y_1,y_2,\ldots,y_q) \mapsto \sum_{j=1}^{q} \alpha_j \delta_{y_j}.$$

Clearly ϕ is a Borel equivariant isomorphism and we set $\kappa = \phi_*(\lambda)$. Then $(M(Y),\kappa)$ is a quasifactor of (Y,ν) which is isomorphic to (Y^q,λ). Thus every self-joining of (Y,ν) is isomorphic to a quasifactor of (Y,ν).

(2) *For $q < \infty$, let $\sigma\colon Y^q \to Y^{(q)} = Y^q/S_q$ be the quotient map, and set $\hat{\lambda} =: \sigma_*(\lambda)$. Define a map*

$$\psi\colon Y^{(q)} \to M(Y),$$

$$\psi\colon \langle y_1,y_2,\ldots,y_q\rangle \mapsto \gamma_{\langle y_1,y_2,\ldots,y_q\rangle} := \frac{1}{q}\sum_{j=1}^{q} \delta_{y_j}.$$

Clearly ψ is a Borel isomorphism and we set $\kappa = \psi_*(\hat{\lambda})$. Then $(M(Y), \kappa)$ is a quasifactor of (Y, ν) which is isomorphic to $(Y^{(q)}, \hat{\lambda})$. It is a joining quasifactor iff the measure λ is symmetric (i.e., λ is invariant under the symmetric group S_q of permutations of coordinates in Y^q). Thus every symmetric q-fold self-joining of (Y, ν) has a $q!$ to 1 factor (an S_q-quotient) which is isomorphic to a joining quasifactor of (Y, ν).

Proof: (1) The barycenter of κ is

$$\begin{aligned} b(\kappa) &= \int \phi(y_1, y_2, \ldots, y_q)\ d\lambda(y_1, y_2, \ldots, y_q) \\ &= \sum_{j=1}^{q} \alpha_j \int \delta_{y_j}\ d\nu(y_j) = \Big(\sum_{j=1}^{q} \alpha_j\Big)\nu = \nu. \end{aligned}$$

Thus $(M(Y), \kappa)$ is a quasifactor of (Y, ν) which is isomorphic to λ.

(2) If λ is symmetric, then the disintegration of λ over $\hat{\lambda}$ is given by

$$\lambda = \int_{Y^{(q)}} \frac{1}{q!} \sum \{\delta_{(y_{\pi(1)}, y_{\pi(2)}, \ldots, y_{\pi(q)})} : \pi \in S_q\}\ d\hat{\lambda}(\langle y_1, y_2, \ldots, y_q \rangle).$$

To check the jqf property we form the measures

$$\kappa' = \int \gamma_{\langle y_1, y_2, \ldots, y_q \rangle} \times \delta_{\langle y_1, y_2, \ldots, y_q \rangle}\ d\hat{\lambda} \quad \text{and} \quad \hat{\kappa} = \int \delta_{y_1} \times \delta_{(y_1, y_2, \ldots, y_q)}\ d\lambda.$$

Now the map

$$\begin{aligned} \mathrm{id} \times \sigma \colon Y \times Y^q &\to Y \times Y^{(q)}, \\ (y, (y_1, y_2, \ldots, y_q)) &\mapsto (y, \langle y_1, y_2, \ldots, y_q \rangle) \end{aligned}$$

is an equivariant map and one can easily check that $(\mathrm{id} \times \sigma)_*(\hat{\kappa}) = \kappa'$. It follows that also κ' is ergodic, so that κ is indeed a joining quasifactor.

On the other hand, if λ is not symmetric, then there exists $\pi \in S_q$ with $\pi_*(\lambda)$ singular to λ and it is easy to see that in this case

$$\kappa' = \int \gamma_{\langle y_1, y_2, \ldots, y_q \rangle} \times \delta_{\langle y_1, y_2, \ldots, y_q \rangle}\ d\hat{\lambda}$$

is not ergodic. ∎

THEOREM 3.6: *Each ergodic system of positive entropy, say $\mathbf{Y} = (Y, \mathcal{Y}, \nu)$, admits every ergodic system of positive entropy, say $\mathbf{X} = (X, \mathcal{X}, \mu)$, as a quasifactor.*

Proof: Since every ergodic system of positive entropy admits a Bernoulli factor of arbitrarily small entropy [S], it is enough, by Lemma 3.1, to prove the assertion for an arbitrary ergodic system $\mathbf{X} = (X, \mathcal{X}, \mu)$ and a Bernoulli system $\mathbf{Y} = (Y, \mathcal{Y}, \nu)$ with entropy $h(\mathbf{Y}) < h(\mathbf{X})$.

By [ST] one can find three Bernoulli factors of $(X, \mathcal{X}, \mu)$, $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$, such that $\mathcal{X} = \bigvee_{j=1}^{3} \mathcal{B}_j$. Let $h_j = h(\mathcal{B}_j), j = 1, 2, 3$, and let $h = \min\{h_1, h_2, h_3\}$. Now if necessary, we split each factor $\mathcal{B}_j$ into a direct product of Bernoulli factors in such a way that we end up with, say, q Bernoulli factors $\mathcal{C}_j, j = 1, \ldots, q$, with $h(\mathcal{C}_j) < h$ and $\mathcal{X} = \bigvee_{j=1}^{q} \mathcal{C}_j$. (When $h(\mu) = \infty$ we will have $q = \infty$.) Next we use the relativized version of Sinai's theorem of [O] to embed each $\mathcal{C}_j$ in a larger Bernoulli factor $\mathcal{A}_j$ with $h(\mathcal{A}_j) = h, j = 1, \ldots, q$. We now have $\mathcal{X} = \bigvee_{j=1}^{q} \mathcal{A}_j$ and, denoting by $\mathbf{Y} = (Y, \mathcal{Y}, \nu)$ the Bernoulli system of entropy h, we see that $\mathbf{X}$ is isomorphic to a q-fold self-joining of $\mathbf{Y}$; $(X, \mu) \cong (Y \times Y \times \cdots \times Y, \lambda)$. Now Lemmas 3.5 and 3.1 complete the proof of the theorem. ∎

PROBLEMS:

(1) *Is this theorem true for joining quasifactors? I.e., does every ergodic system of positive entropy admit every ergodic system of positive entropy as a joining quasifactor?*
(2) *Find an example of two ergodic systems* $\mathbf{X}$ *and* $\mathbf{Y}$ *such that* $\mathbf{Y}$ *appears as a quasifactor but not as a jqf of* $\mathbf{X}$.

A minor contribution toward a solution of these questions is the following observation.

PROPOSITION 3.7: *Let* $\mathbf{X} = (X, \mu)$ *be an ergodic system,* $\mathbf{X} \xrightarrow{\pi} \mathbf{Y} = (Y, \nu)$ *a factor, with* $\mathbf{Y}$ *weakly mixing. Let*

$$\mu = \int_Y \mu_y \, d\nu(y)$$

be the disintegration of μ *over* ν. *Given* $n \geq 1$ *let* $\phi\colon Y^{(n)} \to M(X)$ *be defined by*

$$\langle y_1, y_2, \ldots, y_n \rangle \overset{\phi}{\mapsto} \frac{1}{n} \sum_{j=1}^{n} \mu_{y_j} := \gamma_{\langle y_1, y_2, \ldots, y_n \rangle}.$$

Set $\kappa = \phi_*(\nu^{(n)})$. *Then* $(M(X), \kappa)$ *is a joining quasifactor of* $\mathbf{X}$ *and the map* ϕ *is an isomorphism of* $(Y^{(n)}, \nu^{(n)})$ *onto* $(M(X), \kappa)$.

Proof: Since the family of measures $\{\mu_y\}$ is a.s. pairwise singular, it follows that the equality $\mu_{y_1} + \mu_{y_1} + \cdots + \mu_{y_n} = \mu_{y'_1} + \mu_{y'_1} + \cdots + \mu_{y'_n}$ implies $\mu_{y_j} = \mu_{y'_j}$, $j = 1, \ldots, n$, after a suitable rearrangement of the set $\{y'_1, \ldots, y'_n\}$. Thus the map $\langle y_1, y_2, \ldots, y_n\rangle \overset{\phi}{\mapsto} \frac{1}{n}\sum_{j=1}^n \mu_{y_j}$ is a.s. 1-1. It is easy to see that κ is an ergodic quasifactor of **X**.

Now to check the jqf property we form the measures

$$\kappa' = \int \gamma_{\langle y_1, y_2, \ldots, y_n\rangle} \times \delta_{\langle y_1, y_2, \ldots, y_n\rangle}\, d\nu^{(n)} \quad \text{and} \quad \hat{\kappa} = \int \mu_{\pi(x_1)} \times \delta_{(x_1, x_2, \ldots, x_n)}\, d\mu^n.$$

Since by assumption **Y** is weakly mixing, the measure $\hat{\kappa}$ on $X \times Y^{(n)}$ is ergodic. Now if we write σ for the quotient map from $Y^n \to Y^{(n)}$, then the map

$$\begin{aligned} \mathrm{id} \times (\sigma \circ \pi^n)\colon X \times X^n &\to X \times Y^{(n)}, \\ (x, (x_1, x_2, \ldots, x_n)) &\mapsto (x, \langle \pi(x_1), \pi(x_2), \ldots, \pi(x_n)\rangle) \end{aligned}$$

is an equivariant map and one can easily check that $(\mathrm{id} \times (\sigma \circ \pi^n))_*(\hat{\kappa}) = \kappa'$. It follows that also κ' is ergodic, so that κ is indeed a joining quasifactor. ∎

COROLLARY 3.8: *Let* **X** *be a weakly mixing system of positive entropy. Then* **X** *admits every Bernoulli system as a joining quasifactor.*

Proof: By Sinai's theorem [S] we can find for every positive $k < h(\mu)$ a Bernoulli factor $(X, \mu) \overset{\pi}{\to} (Y, \nu)$ with $h(\nu) = k$. Now for every $n \geq 1$, $Y^{(n)}$ is a Bernoulli system with entropy nk and our corollary follows from Proposition 3.7. ∎

4. Weak Pinsker systems as joining quasifactors

Recall that a system (X, μ) satisfies the **weak Pinsker property** (WP for short), if for every $\delta > 0$ there exist a Bernoulli factor $(X, \mu) \to B$ and an independent factor $(X, \mu) \to Y$ with $h_\mu(Y) = \delta$ such that $(X, \mu) = B \times Y$; see [T] and [Fi]. One of the central open problems in measurable ergodic theory is: do there exist positive entropy systems that are not WP? In view of this, the next theorem represents a significant step in answering the problem that we posed in section 3.

THEOREM 4.1: *If (Y, ν) is a weakly mixing system with positive entropy and (X, μ) an ergodic system with the WP property and with finite positive entropy, then some finite to one group factor of (X, μ) is isomorphic to a joining quasifactor of (Y, ν).*

In proving Theorem 4.1 the following theorem will be used (in place of the Smorodinsky–Thouvenot result ([ST]) that was used in proving Theorem 3.6).

THEOREM 4.2: *If (X,μ) is an ergodic system with the WP property and with positive entropy h_0, then for every $h>0$ there is a Bernoulli system A with entropy $<h$ and a symmetric k-fold selfjoining λ of A for some k, with (A^k,λ) isomorphic to (X,μ).*

In turn, the main tool for the proof of Theorem 4.2 is the next proposition. Let $\{\xi_n : n\in\mathbb{Z}\}$ be an i.i.d. $\{0,1\}$-valued process with

$$P(\xi_n=0)=1-\epsilon,\quad P(\xi_n=1)=\epsilon.$$

We say that a $\{\pm1\}$-valued process $\{\eta_n : n\in\mathbb{Z}\}$ is **resistant to a $(1-\epsilon)$-erasure channel**, if the independent joining $\{(\xi_n,\eta_n) : n\in\mathbb{Z}\}$ can be recovered from the process $\{\zeta_n=\eta_n\,\xi_n : n\in\mathbb{Z}\}$. In other words, there exists a measurable function

$$F(\{\zeta_n\})=\{(\xi_n,\eta_n)\}.$$

PROPOSITION 4.3: *For every $0<\epsilon<1$ and every ergodic system $\mathbf{Y}=(Y,\mathcal{Y},\nu)$ with $h(\mathbf{Y})<\epsilon$, there exists a measurable partition $\{Q_1,Q_{-1}\}$ of Y which is a generator for $\mathbf{Y}$ and for which the $\{\pm1\}$-valued process $\{\eta_n : n\in\mathbb{Z}\}$ defined on $\mathbf{Y}$ by*

$$\eta_n(y)=\mathbf{1}-2\cdot\mathbf{1}_{Q_{-1}}$$

is resistant to a $(1-\epsilon)$-erasure channel.

Proof: As is well known, the $(1-\epsilon)$-erasure channel has positive capacity equal to ϵ (see, for example, [CK] page 114). Being a discrete memoryless channel it satisfies the hypotheses of Theorem 1 in [K] and then our result follows immediately from his corollary.

Proof of Theorem 4.2: Let (X,μ) be a WP system with entropy $h_0>0$. We choose an $\epsilon>0$ such that

$$h_1=H(1-\epsilon,\epsilon/2,\epsilon/2)<h,$$

and such that for some positive integer k and $0<\delta<\min(h_0,\epsilon)$,

$$\begin{aligned} h_0-\delta &= -(1-\epsilon)\log(1-\epsilon)-2^{2k}\left(\frac{\epsilon}{2^{2k}}\right)\log\frac{\epsilon}{2^{2k}}\\ &= -(1-\epsilon)\log(1-\epsilon)-\epsilon\log\epsilon+\epsilon 2k.\end{aligned}$$

Next define a Bernoulli process

$$\{\bar{\xi}_n\}=\{(\xi_n^{(1)},\ldots,\xi_n^{(2k+1)})\}$$

with values in the $2^{2k}+1$ element set

$$\{(\epsilon_1,\dots,\epsilon_{2k+1}) : \epsilon_i = \pm 1,\ \prod_{i=1}^{2k+1} \epsilon_i = 1\} \cup \{(0,\dots,0)\},$$

and entropy $h_0 - \delta$:

$$\bar{\xi}_n = (\xi_n^{(1)},\dots,\xi_n^{(2k+1)}) = \begin{cases} (0,0,\dots,0), & \text{with probability } 1-\epsilon, \\ (\pm 1,\dots,\pm 1), & \text{with probability } \epsilon/2^{2k}. \end{cases}$$

Using the weak Pinsker property we write $X = B \times Y$ with B represented by the Bernoulli process $\{\bar{\xi}_n\}$ and Y an independent factor with entropy δ. Then by an appropriate choice of a partition (Proposition 4.3) $\{Q_1, Q_{-1}\}$, the process $\{\eta_n : n \in \mathbb{Z}\}$ defined on Y by

$$\eta_n(y) = \mathbf{1} - 2\cdot \mathbf{1}_{Q_{-1}}$$

will have the properties:

(1) The process

$$\{(\xi_n^{(1)},\dots,\xi_n^{(2k+1)};\eta_n)\}$$

generates (X,μ), and

(2) the process $\{\eta_n : n \in \mathbb{Z}\}$ is resistant to a $(1-\epsilon)$-erasure channel.

Set

$$\theta_n^{(i)} = \xi_n^{(i)}\eta_n, \quad \bar{\theta}_n = (\theta_n^{(1)},\dots,\theta_n^{(2k+1)}).$$

It is then easy to check that the process $\{\bar{\theta}_n\}$ is identically distributed. Moreover, it can be checked that the process $\{\theta_n^{(i)}\}$ is symmetric in i, hence can be viewed as a symmetric joining of $2k+1$ Bernoulli processes $\{\bar{\theta}_n = \theta_n^{(i)}\}$, each of entropy $h_1 = H(1-\epsilon,\epsilon/2,\epsilon/2)$.

Now set

$$\xi_n = \prod_{i=1}^{2k+1} \xi_n^{(i)},$$

and observe that

$$\xi_n = \begin{cases} 0, & \text{with probability } 1-\epsilon, \\ 1, & \text{with probability } \epsilon, \end{cases}$$

is an erasure channel with probability $1-\epsilon$. Finally, since $\eta_n^{2k+1} = \eta_n$ we get

$$\eta_n\xi_n = \prod_{i=1}^{2k+1} \theta_n^{(i)}.$$

The nature of the partition $\{Q_1, Q_{-1}\}$ implies that the full process

$$\{(\xi_n^{(1)}, \ldots, \xi_n^{(2k+1)}; \eta_n)\}$$

(which generates (X, μ)) is also generated by the process $\{\bar{\theta}_n\}$. The latter is a $(2k+1)$-fold symmetric selfjoining of a Bernoulli system of entropy $h_1 = H(1-\epsilon, \epsilon/2, \epsilon/2) < h$. ∎

Proof of Theorem 4.1: Recall that we are given two ergodic systems with positive entropy (X, μ) and (Y, ν), where (X, μ) is weakly mixing and has the WP property, and we want to find a finite to one group factor of (X, μ) as a joining quasifactor of (Y, ν). By Corollary 3.3 and Proposition 3.4 we can assume that (Y, ν) is Bernoulli with arbitrarily small entropy. An application of Theorem 4.2 and the use of Lemma 3.5(2) complete the proof. ∎

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